Lecture no. 34

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Measure and Integration

28/3/11

$L_p(\mu)$

$(f+g)(x) = f(x) + g(x), \forall x$

$(\alpha f)(x) = \alpha (f(x))$

$\alpha \in \mathbb{C}, f \in L_p(\mu)$

$|\alpha f|^p = |\alpha|^p |f|^p$

$\Rightarrow \int |\alpha f|^p d\mu = |\alpha|^p \int |f|^p d\mu < +\infty$

$\Rightarrow \alpha f \in L_p(\mu).$

$f, g \in L_p(\mu)$

To Show $\int |f+g|^p \, d\mu < +\infty$?

$$|f+g|^p \le (|f|+|g|)^p$$
$$\le \left(2 \max\{|f|,|g|\}\right)^p$$
$$= \cancel{\le}\, 2^p(\max(|f|^p,|g|^p))$$
$$\le 2^p(|f|^p+|g|^p)$$

$$\int |f+g|^p \leq 2^p \left(\int |f|^p d\mu + \int |g|^p d\mu \right)$$

$$< +\infty$$

$\Longrightarrow$ $f, g \in L_p(\mu)$, then

$(f+g) \in L_p(\mu)$

$\alpha \in \mathbb{C},\ f \in L_p$

$$\|\alpha f\|_p = \left(\int |\alpha f|^p \, d\mu\right)^{1/p}$$

$$= \left(|\alpha|^p \int |f|^p \, d\mu\right)^{1/p}$$

$$= |\alpha| \left(\int |f|^p \, d\mu\right)^{1/p}$$

$$= |\alpha| \, \|f\|_p.$$

$$f \in L_p, \quad g \in L_q, \quad \frac{1}{p}+\frac{1}{q}=1$$

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q \; ?$$

<u>Note</u> if $\|f\|_p = 0$ or $\|g\|_q = 0$

$$\Rightarrow \int |f|^p \, d\mu = 0$$

$$\Rightarrow f(x) = 0 \text{ a.e.}$$

$$\Rightarrow fg = 0 \text{ a.e.}$$

$$f(x) = (1-t) + tx - x^t$$

$$f'(x) = t - t\,x^{t-1}$$

$$f'(x) = 0 \Rightarrow t(1 - x^{t-1}) = 0$$

$$\Rightarrow x = 1$$

$$f''(x) = -t(t-1)x^{t-2}$$

$$f''(1) = -t(t-1) > 0$$

$\Rightarrow$ $x = 1$ is a point of local min.

$$f(x) = (1-t) + tx \cancel{\;} - x^t$$

$$f(1) = (1-t) + (t-1)$$
$$= 0.$$

To show
$$f(x) \geq 0 = f(1) \quad \forall x$$

<u>Claim</u> $f(x)$ has minimum at $x = 1$.

$$\int |fg| \, d\mu \le \frac{1}{p} \|f\|_p$$

$$\frac{|fg|}{\|f\|_p^{1/p}\,\|g\|_q^{1/q}} \leq \frac{1}{p}\frac{|f|^p}{\|f\|_p} + \frac{1}{q}\left(\frac{|g|}{\|g\|_q}\right)^q$$

$$\frac{\int |fg|\,d\mu}{\|f\|_p^{1/p}\,\|g\|_q^{1/q}} \leq \frac{1}{p}\,\frac{\|f\|_p^p}{\|f\|_p} + \frac{1}{q}\,\frac{\|g\|_q^q}{\|g\|_q}$$

$$\left(\frac{|f|^p}{\|f\|_p}\right)^{1/p}\left(\frac{|g|^q}{\|g\|_q}\right)^{1-\frac{1}{p}}$$

$$\leq \frac{1}{p}\left(\frac{|f|^p}{\|f\|_p}\right)$$

$$+\left(1-\frac{1}{p}\right)\left(\frac{|g|^q}{\|g\|_q}\right)$$

$\|f\|_p \neq 0$ and $\|g\|_q \neq 0$.

$$t = 1/p, \quad a = \left(\frac{|f|}{\|f\|_p}\right)^p, \quad b = \left(\frac{|g|}{\|g\|_q}\right)^q$$

$$a^t b^{1-t} \leq ta + (1-t)b$$

$$\frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq \frac{1}{p}\left(\frac{|f|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g|}{\|g\|_q}\right)^{1/q}$$

$$\frac{\int |fg|\,d\mu}{\|f\|_p \|g\|_p} \le \frac{1}{p} + \frac{1}{q} = 1$$

$$\int |fg| \le \|f\|_p \|g\|_q.$$

<u>Suppose</u>

$$\|f\|_p \neq 0, \; \|g\|_q \neq 0.$$

$$\boxed{a^t b^{1-t} \leq ta + (1-t)b}$$

for $t = 1/p$, $\cancel{a = \|f\|_p, \; b = \|g\|_q}$

$$a = \frac{|f|^p}{\|f\|_p}, \quad b = \frac{|g|^q}{\|g\|_q}$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$q + p = pq$$

$$p = pq - q$$

$$\boxed{p = (p-1)q}$$

$$|f+g|^{p-1} \in L_q$$

$$\because \int |f+g|^{(p-1)q} \, d\mu$$

$$= \int |f+g|^{p} \, d\mu < +\infty$$